UNIFORMLY NON-$1_n^{(1)}$ ORLICZ SPACES*

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ABSTRACT

A characterisation of uniformly non-$1_n^{(1)}$Orlicz space is obtained intrinsically in terms of the Young function determining the Orlicz space. It is shown that a uniformally non-$1_n^{(1)}$Orlicz space is reflexive.

In a recent paper James [1] conjectured that a uniformly non-$1_n^{(1)}$Banach space is reflexive. Here it is proposed to establish the conjecture when the Banach space is an Orlicz space. We also obtain an intrinsic characterisation of uniformly non-$1_n^{(1)}$ Orlicz spaces.

We start with the basic terminology and definitions required in what follows.

Let (X,S,μ) be a non-atomic measure space and Φ be a non-zero Young function. We adopt the convention $\Phi(u)=\Phi(|u|)$ if u is a real number. The Orlicz set L_Φ is the set of all real valued μ-measurable functions f such that $M(f)=\int_X\Phi(f)d\mu<\infty$. It is known, Weiss [5], that L_Φ is linear if and only if Φ satisfies the growth condition $\Phi(2u)\leqq K\Phi(u)$ for large values of $u(\Phi(2u)\leqq k\Phi(u)$ for all $u\geqq 0)$ if $\mu(X)$ is positive finite (if $\mu(X)$ is infinite). Further the linear space L_Φ can be equipped with a norm $\|\ \|$ by setting

$$\|f\|=\inf\left\{\frac{1}{\xi}\ \Big|\ \xi>0 \text{ and } M(\xi f)\leqq 1\right\}.$$

The normed linear space $(L_\Phi,\|\ \|)$ is indeed a Banach space and we denote this Banach space by L_Φ^*. For a detailed account of this class of Banach spaces we refer to Luxemburg [2], Weiss[5], and Zaanen [6].

REMARK 1. We assume throughout thae paper that Φ satisfies one or the other of the growth conditions ensuring that L_Φ is linear. Thus, in particular $\Phi(u)$ is finite for all real u. Since Φ is convex continuous function it follows that if $u\geqq v>0$ and $\Phi(u)\neq 0$ then $\Phi(u)>\Phi(v)$.

REMARK 2. With regard to the functions M(f) and $\|f\|$ we note that $M(f)\leqq 1$ if and only if $\|f\|\leqq 1$ and $M(f)=1$ if and only if $\|f\|=1$.

Remark 2 is an easy consequence of the definitions.

DEFINITION 1. (JAMES [1]). A normed linear space B is uniformly non-$1_n^{(1)}$ $(n\geqq 2)$ if there exists a positive number δ such that for any n elements $x_1,\cdots,x_n$ in B with $\|x_i\|\leqq 1$ it is true that

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$$\left\| \frac{1}{n}(x_1 \pm x_2 \pm \cdots \pm x_n) \right\| \leqq 1 - \delta$$

for some choice of signs.

REMARK 3. If a positive number δ exists satisfying the above definition then it is clear that $1 > 1 - \delta \geqq 1/n$.

In the case when the normed linear space B is the space L_Φ^* the definition 1 may be reformulated as follows.

DEFINITION 2. L_Φ^* is uniformly non-$1_n^{(1)}$ if for some positive number ξ, $1 < \xi < n$,

$$M\left(\xi \frac{f_1 \pm f_2 \pm \cdots \pm f_n}{n} \right) \leqq 1$$

for some choice of signs where $f_1, \cdots, f_n$ are n functions in L_Φ^* such that $M(f_i) \leqq 1$.

The equivalence of definitions 1 and 2 in the case of Banach spaces L_Φ follows from Remark 2.

With regard to expressions of the form $u_1 \pm u_2 \pm \cdots \pm u_n$ where $\{u_i\}_{i=1}^n$ are n real numbers we adopt the following notation. For a given n-set of reals $\{u_i\}_{i=1}^n$ once for all we enumerate the 2^{n-1} possible expressions of the above form and designate them as $E_1, E_2, \cdots, E_{2^{n-1}}$, and for K, $1 \leqq K \leqq 2^{n-1}$ we denote by U_K the n-vector $(u_1, \pm u_2, \cdots, \pm u_n)$ where the signs in front of the u_i are the same as those occurring in front of u_i in E_K. We define the permutations P_t, $1 \leqq t \leqq n$, of any n-vector $Y = (y_1, \cdots, y_n)$ by setting $P_t Y = (p_t y_1, p_t y_2, \cdots, p_t y_n)$ where $p_t y_i = y_{i+t-1 (\mathrm{mod}\, n)}$ for $1 \leqq i \leqq n$. Further $T_t U_k = \mathrm{sign}_K t P_t U_K$ where sign ${}_K t = 1$ or -1 according as the sign in front of u_t in E_K is $+$ or $-$. We define the functions S and S_1 on the n-dimensional space R^n by setting $S(U) = \sum_{i=1}^n \Phi(u_i)$ and

$$S_1(U; \xi) = \Sigma \Phi \left(\xi \frac{u_1 \pm \cdots \pm u_n}{n} \right)$$

where ξ is a real number, $U = (u_1, \cdots, u_n)$ and the summation in the definition of S_1 is over the 2^{n-1} possible choices of signs. We denote by $C_i U$ the ith coordinate of the vector U.

We proceed now to obtain characterisations of uniformly non-$1_n^{(1)}$ L_Φ^* spaces. We present our characterisations separately in the cases (i) $\mu(X)$ is infinite and (ii) $\mu(X)$ is positive finite.

THEOREM 1. *If (X, S, μ) is an infinite non-atomic measure space then L_Φ^* is uniformly* non-$1_n^{(1)}$ *if and only if there exists a real number $\xi, \xi > 1$, such that*

$$S_1(U; \xi) \leqq \frac{2^{n-1}}{n} S(U) \text{ if } u_i \geqq 0.$$

Proof. Suppose the function Φ satisfies the inequality in the theorem for some $\xi > 1$. Let $\{f_i\}_{i=1}^n$ be n functions in L_Φ^* such that $M(f_i) \leqq 1$. The inequality above clearly implies

$$\Sigma\, M\left(\xi\, \frac{f_1 \pm f_2 \pm \cdots \pm f_n}{n}\right) \leqq \frac{2^{n-1}}{n} \sum_{i=1}^{n} M(f_i) \leqq 2^{n-1}$$

where the summation of the left side of the inequality is over the 2^{n-1} possible choices of signs. Thus there is a choice of signs for which

$$M\left(\xi\, \frac{f_1 \pm f_2 \pm \cdots \pm f_n}{2}\right) \leqq 1.$$

Hence by definition 2, L_Φ^* is uniformly non-$1_n^{(1)}$.

Conversely let L_Φ^* be uniformly non-$1_n^{(1)}$. Let ξ be a real number assured by definition 2 for such a space. If possible let there be a n-vector $U = (u_1, \cdots, u_n)$ with coordinates nonnegative reals such that

$$S_1(U;\xi) > \frac{2^{n-1}}{n} S(U).$$

Since Φ is convex function satisfying the growth condition $\Phi(2u) \leqq K\Phi(u)$ for $u \geqq 0$ there exists a positive number λ such that $\Phi(\xi u) \leqq \lambda\Phi(u)$ for $u \geqq 0$. Hence the inequality above implies $S(U) > 0$. Since (X, S, μ) is an infinite non-atomic measure space there exist 2^{n-1} pairwise disjoint measurable sets $\{A_i\}_{i=1}^{2n-1}$ such that

$$\mu(A_i) = \frac{n}{2^{n-1}\, S(U)} \ . \text{ Let } \{A_i^m\}_{m=1}^n, \text{ for } 1 \leqq i \leqq 2^{n-1},$$

be a measurable partition of A_i such that

$$\mu(A_i^m) = \frac{1}{2^{n-1}\, S(U)} \ .$$

Now we define n functions $\{f_i\}_{i=1}^n$ in L_Φ^* by setting

$$f_i = \sum_{K=1}^{2^{n-1}} \sum_{t=1}^{n} C_i T_t U_K \chi_A t_K .$$

It is verified that

$$M\left(\xi\, \frac{f_1 \pm f_2 \pm \cdots \pm f_n}{n}\right) = \frac{nS_1(U;\xi)}{2^{n-1}S(U)} > 1$$

for any combination of signs while $M(f_i) = 1$ for $1 \leqq i \leqq n$. Thus a contradiction on the choice of ξ is obtained and the proof of the Theorem is complete.

We proceed to the case when $0 < \mu(X) < \infty$. We state first a definition and establish some auxiliary lemmas.

Following the terminology in Nakano [3] the Banach space L_Φ^* is said to be uniformly finite if

$$\sup_{M(f)\leqq 1} M(Kf) < \infty$$

for any positive real number K.

LEMMA 1. *If L_Φ^* is uniformly finite then if $\{f_n\}_{n\geqq 1}$ is a sequence of functions in the unit ball of L_Φ^* such that $\|f_n\| \to 1$ as $n \to \infty$ then $M(f_n) \to 1$ as $n \to \infty$.*

The proof of the lemma is an immediate consequence of Th. 4 on p. 224 in Nakano [3].

LEMMA 2. *If L_Φ^* is uniformly finite and if there exist real numbers t and η $0 < t, \eta < 1$, such that if $\{f_i\}_{i=1}^n$ are any n functions in the unit ball of L_Φ^* with $M(f_i) \geqq 1 - t$ for $1 \leqq i \leqq n$ then for some choice of signs*

$$M\left(\frac{f_1 \pm \cdots \pm f_n}{n}\right) \leqq 1 - \eta$$

L_Φ^* is uniformly non-$1_n^{(1)}$.

Proof. If L_Φ^* is not uniformly non-$1_n^{(1)}$, then for any sequence of reals ξ_j such that $\xi_j > 1$ and $\xi_j \to 1$ there exist n sequences $\{f_i^j\}_{j\geqq 1}$, $1 \leqq i \leqq n$ in the unit ball of L_Φ^* with the property

$$\frac{1}{\xi_j} < \left\| \frac{f_1^j \pm \cdots \pm f_n^j}{n} \right\| \leqq 1$$

for all choices of signs, Thus

$$\lim_{j\to\infty} \left\| \frac{f_1^j \pm \cdots \pm_n^j}{n} \right\| \to 1$$

for all choices of signs. Since $\|f_i^j\| \leqq 1$ each of the sequences $\{f_i^j\}_{j\leqq 1}$ admits a subsequence $\{g_i^j\}_{j\leqq 1}$ $1 \leqq i \leqq n$, such that $\lim_{j\to\infty} \|g_i^j\| \to 1$ for i, $1 \leqq i \leqq n$, and

$$\lim_{j\to\infty} \left\|\frac{g_1^j \pm \cdots \pm g_n^j}{n}\right\| \to 1$$

for all choices of signs. Since L_Φ^* is uniformly finite, by Lemma 1 $M(g_i^j) \to 1$ as $j \to \infty$ *for* $1 \leqq i \leqq n$, and

$$\lim_{j\to\infty} M\left(\frac{g_i^j \pm \cdots \pm g_n^j}{n}\right) = 1$$

for all choices of signs, a contradiction on the inequality in the hypothesis.

LEMMA 3. *If (X,S,μ) is a positive finite measure space and if there are two positive numbers λ, ξ and v with $\xi > 1$ such that $\Phi(\xi u) \leqq \lambda\Phi(u)$ for $u \geqq v>0$ then L_Φ^* is uniformly finite.*

Proof. Let K be a positive number. If f is in the unit ball of L_Φ^* and t is a positive integer such that $\xi^t > K$ and if

$$\begin{aligned} E &= \{x \mid x\in X,\ |f(x)| \leqq v\} \text{ then} \\ M(Kf) &\leqq M(\xi^t f) \\ &= \int_E \Phi(\xi^t f)d\mu + \int_{X\sim E} \Phi(\xi^t f)d\mu \\ &\leqq \Phi(\xi^t v)\mu(E) + \lambda^t M(f) \\ &\leqq \Phi(\xi^t v)\,\mu(X) + \lambda^t. \end{aligned}$$

Thus $\sup_{M(f)\leqq 1} M(Kf) < \infty$. Hence L_Φ^* is uniformly finite.

THEOREM 2. *If (X,S,μ) is a positive finite non-atomic measure space then L_Φ^* is uniformly* non-$l_n^{(1)}$ *if and only if*

(i) *There exist positive numbers C and ξ, $1 < \xi$, such that*

$$S_1(U;\xi) \leqq \frac{2^{n-1}}{n} S(U) \text{ if } S(U) \geqq C > 0 \text{ and } C_i U \geqq 0 \text{ for } 1 \leqq i \leqq n.$$

(ii) $$\Phi\left(\frac{v_0}{n}\right) \neq \frac{1}{n}\Phi(v_0) \text{ if } \Phi(v_0) = \frac{n}{\mu(X)}.$$

Proof. Let us assume that L_Φ^* is uniformly non-$l_n^{(1)}$. Then definition 2 guarantees the existence of real number $\xi_0 > 1$ such that if $\{f_i\}_{i=1}^n$ are any n functions in the unit ball of L_Φ^* then for some choice of signs

$$M\left(\xi_0 \frac{f_1 \pm \cdots \pm f_n}{n}\right) \leqq 1.$$

We shall prove that Φ fulfills the condition (i) with the choice of $\xi = \xi_0$. Assuming the contrary there exist non-negative numbers $\{u_i\}_{i=1}^n$ such that

$$S(U) \geqq \frac{n}{\mu(X)} \text{ and } S_1(U;\xi) > \frac{2^{n-1}}{n} S(U).$$

Since

$$\mu(X) \geqq \frac{n}{S(U)}$$

there exist 2^{n-1} pairwise disjoint measurable sets $\{A_i\}_{i=1}^{2n-1}$ such that

$$\mu(A_i) = \frac{n}{2^{n-1}S(U)} \text{ for } 1 \leqq i \leqq 2^{n-1}.$$

Now we can complete the proof by constructing the functions $\{f_i\}_{i=1}^n$ as in theorem 1 contradicting the choice of ξ_0. Next we shall show that

$$\Phi\left(\frac{v_0}{n}\right) \neq \frac{1}{n}\Phi(v_0).$$

If possible let

$$\Phi\left(\frac{v_0}{n}\right) = \frac{1}{n}\Phi(v_0). \text{ Since } \Phi(v_0) = \frac{n}{\mu(X)}$$

and X is a nonatomic measure space there exist n pairwise disjoint measurable sets $\{A_i\}_{i=1}^n$ such that $\Phi(v_0)\mu(A_i) = 1$. Let $f_i = v_0\chi_{A_i}$ for $1 \leqq i \leqq n$. Then clearly $M(f_i) = \|f_i\| = 1$ for $1 \leqq i \leqq n$ and for all choices of signs

$$M\left(\xi_0 \frac{f_1 \pm f_2 \pm \cdots \pm f_n}{n}\right) = \sum_{i=1}^n \Phi\left(\frac{v_0}{n}\xi_0\right)\mu(A_i)$$

$$> \sum_{i=1}^n \Phi\left(\frac{v_0}{n}\xi_0\right)\mu(A_i) \geqq 1$$

since

$$\Phi\left(\frac{v_0}{n}\xi_0\right) > \Phi\left(\frac{v_0}{n}\right)$$

by Remark 1. Hence a contradiction arises on the choices of ξ_0.

Next we shall prove that if Φ satisfies the inequalities (i) and (ii) then L_Φ^* is uniformly non-$1_n^{(1)}$. If Φ satisfies inequality (i) and $\Phi(v_1) = C$ then by choosing the vector U such that $C_iU = u \geqq v_1$ it is verified that $\Phi(\xi u) \leqq 2^{n-1}\Phi(u)$ for all $u \geqq v_1$. Thus L_Φ^* is uniformly finite by Lemma 3. Since

$$\Phi\left(\frac{v_0}{n}\right) \neq \frac{1}{n}\Phi(v_0)$$

and Φ is continuous it follows by Remark 1 that there exists a real number v_1 such that $0 < v_1 < v_0$, $\Phi(v_1) > 0$, and

$$\Phi\left(\frac{v_1}{n}\right) \neq \frac{1}{n}\Phi(v_1) \neq \frac{1}{n}\Phi(v_0). \text{ Let } \Phi(v_1) = \theta\,\Phi(v_0) = \theta\frac{n}{\mu(X)}$$

where $0 < \theta < 1$. In the condition (i) we can assume that $C \geqq n/\mu(X)$.

Let

$$K = \left\{U \,\middle|\, U \in R^n \text{ and } \frac{\theta n}{\mu(X)} \leqq S(U) \leqq C\right\}.$$

We note that $S_1(U;1) < (2^{n-1}/n)\,S(U)$ for $U \in K$ as a consequence of Remark 1 and of the inequality $\Phi(v_1/n) < (1/n)\,\Phi(v_1)$. Since Φ is continuous and K is a compact subset of R^n there exists a real number $\xi_1 > 1$ such that

$$S_1(U;\xi_1) < \frac{2^{n-1}}{n} S(U) \text{ for all } U \in K.$$

Let $\xi_0 = \min(\xi_1, \xi)$. Then for all U such that

$$S(U) \geqq \frac{\theta\eta}{\mu},\ S_1(U;\xi_0) < \frac{2^{n-1}}{n} S(U).$$

Let t be a real number such that $1 - t > \theta$. We shall prove that there exists an η, $1 > \eta > 0$ such that if $\{f_i\}_{i=1}^n$ are n functions in the unit ball of L_Φ^* with

$$M(f_i) \geqq 1 - t \text{ for } 1 \leqq i \leqq n$$

then for some choice of signs

$$M\left(\frac{f_1 \pm f_2 \pm \cdots \pm f_n}{n}\right) \leqq 1 - \eta.$$

Let $E = \{x \mid \sum_{i=1}^n \Phi(f_i(x)) \geqq \theta\eta/\mu$. Then we obtain the following inequality.

$$\begin{aligned}\sum M\left(\frac{f_1 \pm f_2 \pm \cdots \pm f_n}{n}\right) &= \sum \int_E \Phi\left(\frac{f_1 \pm f_2 \pm \cdots \pm f_n}{n}\right) d\mu \\ &\quad + \sum \int_{X \sim E} \Phi\left(\frac{f_1 \pm \cdots \pm f_n}{n}\right) d\mu \\ &\leqq \frac{1}{\xi_0}\frac{2^{n-1}}{n} \sum \int_E \Phi(f_i) d\mu + \frac{2^{n-1}}{n} \int_{X \sim E} \Phi(f_i) d\mu \\ &\leqq 2^{n-1} - \left(1 - \frac{1}{\xi_0}\right) 2^{n-1} (1 - t - \theta) \cdots (1)\end{aligned}$$

since

$$\begin{aligned}\sum \int_E \Phi(f_i) d\mu &\geqq n(1-t) - \frac{\theta\eta}{\mu(X)} \mu(X \sim E) \\ &\geqq n(1-t) - n\theta.\end{aligned}$$

Setting $1 - \eta = (1 - t - \theta)(1 - 1/\xi_0)$ we obtain from inequality (1) that for some choice of signs

$$M\left(\frac{f_1 \pm f_2 \pm \cdots \pm f_n}{n}\right) \leqq 1 - \eta.$$

Hence L_Φ^* is uniformly finite and the associated function $M(f)$ satisfies the inequality in Lemma 2. Thus L_Φ^* is uniformly non-$1_n^{(1)}$ and the proof of Theorem 2 is complete.

James [1] has shown that a uniformly non-$1_n^{(1)}$ Banach space is reflexive if it has an unconditional basis. Since it is not assumed that μ is separable, the Banach space L_Φ^* is not necessarily separable, Luxemburg [2]. However, it will be shown that every uniformly non-$1_n^{(1)}$ Orlicz space L_Φ^* is reflexive. We observe first a consequence of the inequalities in the statements of Theorems 1 and 2. If for some $\xi > 1$,

$$S_1(U;\xi) \leqq \frac{2^{n-1}}{n} S(U)$$

for every n-vector U with non-negative coordinates then choosing U such that $C_1 U = u \geqq 0$ and $C_i U = 0$ for $2 \leqq i \leqq n$ it is verified that (A) for $u \geqq 0$,

$$\Phi(\frac{u}{n}\xi) \leqq \frac{1}{n}\Phi(u).$$

Similarly if

$$S_1(U;\xi) \leqq \frac{2^{n-1}}{n} S(U)$$

for U such that $S(U) \geqq C > 0$ then there exists a $v > 0$ such that (B) for all $u \geqq v > 0$,

$$\Phi\left(\frac{u}{n}\xi\right) \leqq \frac{1}{n}\Phi(u).$$

If Ψ is the Young's complement of Φ, and if Φ satisfies (A) or (B) then there exists a constant $K > 0$ such that $\Psi(2u) \leqq K\Psi(u)$ for all $u \geqq 0$ or $\Psi(2u) \leqq K\Psi(u)$ for large values of u according as (A) or (B) is true.

THEOREM 3. *If L_Φ^* is uniformly* non-$1_n^{(1)}$ *then it is reflexive.*

Proof. Since Φ and Ψ satisfy the growth conditions ensuring that L_Φ and L_ψ are linear it follows from theorems 4 and 5, Rao [4] that L_Φ^* is reflexive.

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